Announcements

1) Scholarships – application deadline is tomorrow (Dept. Website)

2) New HW on CTools, due Thursday

Chapter 4

Continuity

Not all functions are nice! How bad can the set of discontinuties of a function be?

Can the discontinuities be "arbitrary"?

Section 4.2

Limits for functions

You've seen this before but...

now we can define functions on odd subsets of the real numbers – like the Cantor set!

Definition: (limit)

Recall: If $S \subseteq \mathbb{R}$, then if $x \in S$, either $x \in S'$ or x is an isolated point of S (but not both).

If $x \in S'$, then if $f : S \cup S' \to \mathbb{R}$, we say f has a limit at x if $\exists\, L \in \mathbb{R}$ such that $\forall \varepsilon > 0$, $\exists\, \delta > 0$ such that

$$|f(y) - L| < \varepsilon$$

when

$$0 < |x - y| < \delta$$

Remember!

$x = y$ is unimportant for the limit - we just care about what happens close to x.

Example 1: (x, piecewise)

Let $f(x) = \begin{cases} 1, & x = 0 \\ x, & x \neq 0, \end{cases}$

Then what should be the limit of f at $x = 0$?

$L = 0$.

If we think $L = 0$, then we have to show:

$\forall \varepsilon > 0, \exists \delta > 0$

such that

$$|f(x)| < \varepsilon \text{ when}$$

$$0 < \underbrace{|x - 0|}_{= |x|} < \delta \quad (0 < |x| < \delta).$$

When $x \neq 0$, $f(x) = x$,

<u>So</u> We want, for every $\varepsilon > 0$, a $\delta > 0$ such that

$$|x| < \varepsilon \text{ when}$$

$$0 < |x| < \delta.$$

$$\delta = \varepsilon$$

Proposition: (sequential)

Let $S \subseteq \mathbb{R}$ and let $f : \overline{S} \to \mathbb{R}$.

Then if $x \in S'$, the limit of f at x is equal to L if and only if for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq S$, $x_n \neq x$ for all $n \in \mathbb{N}$,

$$\lim_{n\to\infty} f(x_n) = L \text{ whenever } x_n \to x.$$

proof: $\Rightarrow$ Suppose the limit of f at x is equal to L. Let $(x_n)_{n\in\mathbb{N}}$ be any sequence in S, $\lim_{n\to\infty} x_n = x$, $x_n \neq x$ $\forall n \in \mathbb{N}$.

Show: $\lim_{n\to\infty} f(x_n) = L$. So $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ with $|f(x_n) - L| < \varepsilon$ for all $n \geq N$.

But $\exists$ $\delta > 0$ such that

$|f(y) - L| < \varepsilon$ whenever

$|y - x| < \delta$.

Then since $\lim_{n \to \infty} x_n = x$,

choose $N \in \mathbb{N}$ so that

$|x_n - x| < \delta$ for all $n \geq N$.

Automatically, we have

$|f(x_n) - L| < \varepsilon$.

$\Leftarrow$ Suppose for all sequences $(x_n)_{n\in\mathbb{N}} \subseteq S$, $x_n \neq x$,

$\lim_{n\to\infty} x_n = x$, then

$\lim_{n\to\infty} f(x_n) = L$.

By contradiction suppose the limit of f at x is not L.

This means $\exists\, \varepsilon > 0$ such that "no $\delta > 0$ works".

So $\forall\ \delta > 0$, $\exists\ y$ with $0 < |x - y| < \delta$, but

$$|f(y) - L| \geq \varepsilon.$$

Fix this $\varepsilon > 0$. Choose a sequence of points $(x_n)_{n \in \mathbb{N}}$ with $0 < |x - x_n| < \frac{1}{n}$ and such that $|f(x_n) - L| \geq \varepsilon$ for all $n \in \mathbb{N}$.

Then $\lim_{n \to \infty} x_n = x$, $x_n \neq x$ for all $n \in \mathbb{N}$, so by our assumption, $\exists\, N \in \mathbb{N}$ with $|f(x_n) - L| < \varepsilon$ for all $n \geq N$, contradiction since we supposed $|f(x_n) - L| \geq \varepsilon$ $\forall\, n \in \mathbb{N}$.

Therefore, the limit of f at x is L $\square$

Notation: If the limit of f at x is equal to L, write

$$\boxed{\lim_{y \to x} f(y) = L}$$

Example 2 (Dirichlet)

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

Then $\lim_{y \to x} f(x)$ does not exist for any $x \in \mathbb{R}$!

Density of rationals and irrationals

If $x \in \mathbb{R}$, choose $(x_n)_{n \in \mathbb{N}}$ with $x_n \in \mathbb{Q}$, $x_n \neq x$, $\lim_{n \to \infty} x_n = x$. Then choose $(y_n)_{n \in \mathbb{N}}$, $y_n \notin \mathbb{Q}$, $y_n \neq x$, $\lim_{n \to \infty} y_n = x$.

Then $f(x_n) = 1 \quad \forall n \in \mathbb{N}$

and $f(y_n) = 0 \quad \forall n \in \mathbb{N}$

So the sequential property will not hold for any choice of L!